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# Random Ising model with long-range interactions

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**Abstract.** The critical properties of a random Ising model with long-range isotropic interactions decaying as $R^{-(d+\sigma)}$ are analysed by using renormalisation group methods in an expansion in $\varepsilon' = 2\sigma - d$, where $d$ is the dimensionality. For $\varepsilon' > 0$ the critical behaviour is described by a stable fixed point $O(\sqrt{\varepsilon'})$ in the physical region of parameter space. The crossover to short-range behaviour is analysed and takes place when $\sigma = 2 + \varepsilon/106$, $\varepsilon = 4 - d > 0$. There is then a small region $0 < \sigma - 2 < O(\varepsilon)$ where the system is still dominated by long-range behaviour. The results for $d = 1$ are compared with those obtained previously for the random hierarchical model and it is found that the two models do not show the same critical behaviour in the $\varepsilon'$ expansion.

## 1. Introduction

The critical behaviour of random $m$-component spin systems with long-range isotropic interactions has been studied by Yamazaki (1978) by means of the Callan–Symanzik equations in the general case $m > 1$. The long-range potential in consideration decays as $R^{-(d+\sigma)}$, $0 < \sigma < 2$, and critical properties are derived in an expansion in $\varepsilon' = 2\sigma - d$ (Fisher *et al* 1972, Yamazaki and Suzuki 1977).

Although random dipolar magnets have been studied both for $m > 1$ and $m = 1$ (Aharony 1975, 1976a), the case of an Ising system with random long-range isotropic interactions has not previously been discussed. It is expected in this case to find a random fixed point $O(\sqrt{\varepsilon'})$, as it is obtained for short-range interactions in an expansion in $\varepsilon = 4 - d$ (Aharony 1976a, Grinstein and Luther 1976, Khmelnitskii 1975, Jayaprakash and Katz 1977). In two previous publications I analysed (Theumann 1980a, b) the critical properties of the hierarchical model (Baker 1972) with random couplings. This is a model for Ising spins in one dimension ($m = 1$, $d = 1$) that in the uniform (non-random) case has analogous critical behaviour as the power law potential $R^{-(1+\sigma)}$, $0 < \sigma < 1$. It was found within an expansion in $\varepsilon' = 2\sigma - 1$ that the random hierarchical model does not have a stable fixed point in the physical region of parameter space for $\varepsilon' > 0$. A fixed point $O(\sqrt{-\varepsilon'})$ exists only for $\varepsilon' < 0$ and it is unstable. In the second paper this 'runaway' was shown to correspond to a spin–glass transition. Given the analogous critical properties of the uniform hierarchical model and the Ising system with power law interactions when $d = 1$ (Kim and Thompson 1977), I considered it of interest to verify whether a similar 'runaway' occurs for *any* $d$, $\sigma < d < 2\sigma$, and $\sigma < 2 - \eta_{SR}$. Here $\eta_{SR}$ stands for the value of the exponent $\eta$ for short-range interactions (Sak 1973). The results reported in this paper show that this is not the case.

In § 2 I introduce the starting Hamiltonian for a random Ising model with power law interactions, and recursion relations are derived in an expansion in $\varepsilon' = 2\sigma - d$ for

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$\sigma < 2 - \eta_{SR}$, by using the $n = 0$ limit (Emery 1975). Here I take both the short- (SR) and long- (LR) range $\boldsymbol{k}$ dependence in the propagator explicitly into account to analyse the crossover to SR behaviour (Sak 1973). Many of the derivations are by now standard in the renormalisation group theory of random systems; therefore I only quote the main results for completeness.

The results are stated in § 3. It is found that for $\varepsilon' > 0$ the transition is described by a stable random fixed point $O(\sqrt{\varepsilon'})$ in the physical region when $\sigma < 2 - \eta_{SR}$. When $\sigma = 2 - \eta_{SR}$ the system crosses over smoothly to SR behaviour. The only peculiarity is that for the random fixed point $\eta_{SR} < 0$ (Aharony 1976a); and it follows from an analysis similar to Sak's (1973) that there is a small region $2 < \sigma < 2 + \varepsilon/106$ where the system still has LR behaviour ($\varepsilon = 4 - d$).

As these results are valid for any dimensionality $d$, the conclusion is that the random hierarchical model (Theumann 1980a, b) does not have the same critical behaviour as the random Ising system with power law interactions in one dimension, within the $\varepsilon'$ expansion.

## 2. Hamiltonian and recursion relations

Many of the derivations involved in this work are by now standard in the renormalisation group theory of random systems; therefore I shall only quote the main results for completeness and refer the interested reader to the original papers.

The starting point is the Hamiltonian for an Ising model with random interactions and continuous spins:

$$\mathcal{H} = -\frac{H}{KT} = \sum_{i,j} J_{ij} S_i S_j + u \sum_i S_i^4 \tag{2.1}$$

where $i, j$ refer to sites in a $d$-dimensional lattice. The couplings $J_{ij}$ can be considered to be independent random bonds with mean and variance

$$\langle J_{ij} \rangle = V(|\boldsymbol{R}_i - \boldsymbol{R}_j|) \tag{2.2}$$

$$\langle J_{ij} J_{lm} \rangle - \langle J_{ij} \rangle \langle J_{lm} \rangle = \delta_{il} \delta_{jm} \Delta_{ij}. \tag{2.3}$$

The configurational averaged free energy can be written (Emery 1975) as

$$-\frac{F}{KT} = \langle \ln Z \rangle = \lim_{n \to 0} \frac{1}{n} [\langle Z^n \rangle - 1]$$

and we have to consider the effective partition function for the $n$-component system $\langle Z^n \rangle$. In momentum space and in the continuum limit the effective Hamiltonian is

$$\begin{aligned}\mathcal{H}_{\mathrm{eff}} = \frac{1}{2} \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d} V(\boldsymbol{k}) \sum_{\alpha=1}^{n} S_\alpha(\boldsymbol{k}) S_\alpha(-\boldsymbol{k}) + u K_d^{-1} \int \frac{\mathrm{d}\boldsymbol{k}_1}{(2\pi)^d} \cdots \frac{\mathrm{d}\boldsymbol{k}_4}{(2\pi)^d} (2\pi)^d \delta(\boldsymbol{k}_1 + \ldots + \boldsymbol{k}_4) \\ \times \sum_{\alpha=1}^{n} S_\alpha(\boldsymbol{k}_1) S_\alpha(\boldsymbol{k}_2) S_\alpha(\boldsymbol{k}_3) S_\alpha(\boldsymbol{k}_4) \\ - \Delta K_d^{-1} \int \frac{\mathrm{d}\boldsymbol{k}_1}{(2\pi)^d} \cdots \frac{\mathrm{d}\boldsymbol{k}_4}{(2\pi)^d} (2\pi)^d \delta(\boldsymbol{k}_1 + \ldots + \boldsymbol{k}_4) \\ \times \sum_{\alpha,\beta=1}^{n} S_\alpha(\boldsymbol{k}_1) S_\alpha(\boldsymbol{k}_2) S_\beta(\boldsymbol{k}_3) S_\beta(\boldsymbol{k}_4)\end{aligned} \tag{2.4}$$

with

$$V(\boldsymbol{k}) = r + ak^2 + gk^\sigma. \tag{2.5}$$

Here $V(\boldsymbol{k})$ is the Fourier transform of $V(\boldsymbol{R})$ in equation (2.3) and the terms $ak^2$, $gk^\sigma$ correspond to SR and LR interactions, respectively. The momentum space integrals are in $d$ dimensions and they have a cut-off $|\boldsymbol{k}| \leq 1$. The numerical factor $K_d$ is:

$$K_d = 2^{1-d}\pi^{-d/2}[\Gamma(d/2)]^{-1} \tag{2.6}$$

and it was introduced for convenience in the definitions of $u$ and $\Delta$. The recursion relations obtained from equations (2.4)–(2.5) after integrating over spin variables with $b^{-1} < |k| < 1$ and rescaling (Fisher 1974, Aharony 1976b) were derived for SR interactions ($g = 0$) by Aharony (1976a) and in the uniform case ($\Delta \equiv 0$) by Sak (1973). In the limit $n = 0$ they become

$$a' = b^{-\eta}[a + 32(2\Delta^2 + 3u^2 - 6u\Delta)(a+g)I_5] \tag{2.7}$$

$$g' = b^{2-\eta-\sigma}g \tag{2.8}$$

$$r' = b^{2-\eta}[r + 4(3u - 2\Delta)(I_1 - rI_2) - 32(2\Delta^2 + 3u^2 - 6u\Delta)I_4(0)] \tag{2.9}$$

$$u' = b^{(4-d-2\eta)}[u - 12I_2(3u^2 - 4u\Delta) + 16(I_2)^2(27u^3 - 54u^2\Delta + 36u\Delta^2) + 32I_3(27u^3 - 72u^2\Delta + 42u\Delta^2)] \tag{2.10}$$

$$\Delta' = b^{(4-d-2\eta)}[\Delta + 8I_2(4\Delta^2 - 3u\Delta) + 16(I_2)^2(20\Delta^3 - 36\Delta^2 u + 27\Delta u^2) + 32I_3(22\Delta^3 - 36\Delta^2 u + 9\Delta u^2)]. \tag{2.11}$$

By calling the propagator at the critical point ($r = 0$)

$$G(\boldsymbol{q}) = (aq^2 + gq^\sigma)^{-1} \tag{2.12}$$

the integrals $I_l$, $l = 1, \ldots, 5$, are given by

$$I_1 = \int_{b^{-1}}^1 q^{d-1}\, \mathrm{d}q\, G(\boldsymbol{q}) \tag{2.13}$$

$$I_2 = \int_{b^{-1}}^1 q^{d-1}\, \mathrm{d}q\, G^2(\boldsymbol{q}) \tag{2.14}$$

$$I_3 = 2K_d^{-2}(2\pi)^{-2d} \iint_> \mathrm{d}\boldsymbol{q}_1\, \mathrm{d}\boldsymbol{q}_2\, G(\boldsymbol{q}_1)G(\boldsymbol{q}_2)G^2(\boldsymbol{q}_1 + \boldsymbol{q}_2) \tag{2.15}$$

$$I_4(k) = K_d^{-2}(2\pi)^{-2d} \iint_> \mathrm{d}\boldsymbol{q}_1\, \mathrm{d}\boldsymbol{q}_2\, G(\boldsymbol{q}_1)G(\boldsymbol{q}_2)G(\boldsymbol{q}_1 + \boldsymbol{q}_2 + \boldsymbol{k}) \tag{2.16}$$

$$I_5 = -\frac{1}{2}\frac{1}{(a+g)}\left(\frac{\partial^2 I_4(k)}{\partial k^2}\right)_{k=0}. \tag{2.17}$$

The integrals cannot be performed in closed form keeping both $a$ and $g$ different from zero; therefore as a first step we repeat the analysis of Sak (1973) and look for the stable fixed point values $a^*$, $g^*$ from equations (2.7)–(2.8).

We find that for:

(i) the LR region

$$g = g^* = 1 \qquad \eta_{\mathrm{LR}} = 2 - \sigma \qquad (b^\eta - 1 - \phi(\eta)I_5)a^* = g^*\phi(\eta)I_5. \tag{2.18}$$

Here $a$ is 'irrelevant' and can be set equal to zero as long as

$$\eta_{LR} > \phi(\eta_{LR}) I_5(\ln(b))^{-1}$$

where

$$\phi(\eta) = 32(2\Delta^{*2} + 3u^{*2} - 6u^*\Delta^*). \tag{2.19}$$

(ii) the SR region

$$a = a^* = 1 \qquad g^* = 0 \qquad \eta_{SR} = \phi(\eta_{SR}) I_5(\ln(b))^{-1} \tag{2.20}$$

stable for

$$\sigma > 2 - \eta_{SR} \tag{2.21}$$

The function $\phi(\eta)$ in equation (2.19) depends on $\eta$ through the solutions $u^*$, $\Delta^*$ of equations (2.10)–(2.11). For the Ising random fixed point $O(\sqrt{\varepsilon'})$ it is necessary to take this dependence into account.

According to these results, the integrals $I_l$, $l = 1, \ldots, 5$ can now be performed by setting $a = 0$, $g = 1$ in the LR region, $\sigma < 2 - \eta_{SR}$, $d = 2\sigma - \varepsilon'$, and they take continuously their SR value when $\sigma = 2$, $d = 4 - \varepsilon$. Because we are interested only in their asymptotic values for large $b$ and the propagator is taken at the critical point, these integrals are simpler than those appearing in the $1/m$ expansion for long-range interactions (Ma 1973, Theumann 1975). Most of them have been previously evaluated by other authors and we give the results in the appendix for completeness.

## 3. Results and conclusions

I start by considering the long-range region, $\sigma < 2 - \eta_{SR}$ and $\eta_{LR} = 2 - \sigma$; then

$$4 - d - 2\eta_{LR} = 2\sigma - d = \varepsilon' \tag{3.1}$$

and the integrals in the recursion relations (2.9)–(2.11) are evaluated at $d = 2\sigma$.

The fixed points of the recursion relations for $u$ and $\Delta$ are:

(*a*) gaussian: $u^* = \Delta^* = 0$, stable for $\varepsilon' < 0$, classical exponents,

(*b*) 'uniform': $\Delta^* = 0$, $u^* = \varepsilon'/36 + O(\varepsilon'^2)$, unstable,

(*c*) 'unphysical': $u^* = 0$, $\Delta^* = -\varepsilon'/32$, stable but inaccessible,

(*d*) random: $\varepsilon' > 0$, stable,

$$u^* = \tfrac{1}{18}(\varepsilon' A^{-1}(\sigma))^{1/2} + O(\varepsilon') \qquad \Delta^* = \tfrac{1}{24}(\varepsilon' A^{-1}(\sigma))^{1/2} + O(\varepsilon') \tag{3.2}$$

where $A(\sigma)$ is given by equation (A6). The value of the exponent $\nu$ at this fixed point is

$$\nu = \frac{1}{\sigma} + \frac{1}{3\sigma^2}(\varepsilon' A^{-1}(\sigma))^{1/2} + O(\varepsilon').$$

The crossover to the SR region occurs when $\sigma = 2 - \eta_{SR}$; then from equation (3.1):

$$\varepsilon' = 4 - d - 2\eta_{SR} = \varepsilon - 2\eta_{SR} \tag{3.3}$$

where $\eta_{SR}$ is the solution of equation (2.20) when $I_5$ and $A(\sigma)$ are evaluated at $\sigma = 2$; $\phi(\eta_{SR})$ is given by equations (2.19) and (3.2). The SR values of $u^*$, $\Delta^*$ and the exponents are, from equations (3.2) and (3.3):

$$u^* = \tfrac{1}{3}(\tfrac{3}{106}\varepsilon)^{1/2} + \mathrm{O}(\varepsilon) \qquad \Delta^* = \tfrac{1}{4}(\tfrac{3}{106}\varepsilon)^{1/2} + \mathrm{O}(\varepsilon)$$
$$\eta_{\mathrm{SR}} = -\tfrac{1}{106}\varepsilon + \mathrm{O}(\varepsilon^{3/2}) \qquad \nu = \tfrac{1}{2} + \tfrac{1}{2}(\tfrac{3}{106}\varepsilon)^{1/2} + \mathrm{O}(\varepsilon) \tag{3.4}$$

in agreement with previous calculations (Aharony 1976a).

It is then found that an Ising system with random LR isotropic interactions has a second-order transition described by a stable fixed point $\mathrm{O}(\sqrt{\varepsilon'})$, $\varepsilon' = 2\sigma - d > 0$. The critical exponents go continuously from their LR $\sigma$-dependent values to their SR values when $\sigma = 2 - \eta_{\mathrm{SR}}$.

As a consequence of the negative value of $\eta_{\mathrm{SR}}$ in equation (3.4), the crossover between LR and SR behaviour occurs for $\sigma = 2 + \varepsilon/106$. Hence a small region $\mathrm{O}(\varepsilon)$ exists where $\sigma > 2$ and the properties of the system are still dominated by LR interactions. For $\varepsilon' < 0$ (LR) and $\varepsilon < 0$ (SR) the transition is the same as in the pure system with classical exponents.

A comparison with the results obtained previously (Theumann 1980a) for the random hierarchical model (RHM) is a bit surprising. The uniform hierarchical model (Baker 1972) shows the same critical behaviour to first order in an expansion in $\varepsilon' = (2\sigma - 1)$ (Bleher and Sinai 1975, Kim and Thompson 1977) as the analogous Ising system with power law interaction $R^{-(1+\sigma)}$ in one dimension (Fisher *et al* 1972), thus it was expected that the same should happen in the random case. However, the RHM does not have a stable fixed point in the physical region, in an expansion in $\varepsilon'$, while the fixed point $\mathrm{O}(\sqrt{|\varepsilon'|})$ only exists for $\varepsilon' < 0$ and is unstable (cf equation (3.2)). It was shown in a later paper that this 'runaway' indicates a spin–glass transition (Theumann 1980b). The reason for this discrepancy is that to obtain the fixed point $\mathrm{O}(\sqrt{|\varepsilon'|})$ one has to work out recursion relations for $u'$ and $\Delta'$ including terms $\mathrm{O}(u^3)$, $\mathrm{O}(u^2\Delta)$, etc as shown in equations (2.10) and (2.11). Now, in the uniform system ($\Delta \equiv 0$), precisely these terms give corrections $\mathrm{O}(\varepsilon'^2)$ to the critical exponents, and the values of the critical exponents for the hierarchical model (Kim and Thompson 1977) coincide to $\mathrm{O}(\varepsilon')$ but differ to $\mathrm{O}(\varepsilon'^2)$ from those for an Ising system with power law interactions (Fisher *et al* 1972). In fact, it should be mentioned that different procedures to carry out the $\varepsilon'$ expansion in the hierarchical model give different results for the correction $\mathrm{O}(\varepsilon'^2)$, while they all coincide to $\mathrm{O}(\varepsilon')$, as discussed by Kim and Thompson (1977).

The procedure I followed to analyse the RHM would give for a uniform system yet another different value for the corrections $\mathrm{O}(\varepsilon'^2)$. None of these coincides with the results of Fisher *et al* (1972).

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## Appendix

The integrals $I_l$, $l = 1, \ldots, 5$ in equations (2.13)–(2.17) are evaluated in the LR region for $a = 0$, $g = 1$, $d = 2\sigma - \varepsilon'$. Their SR value is obtained by making $\sigma = 2$, $d = 4 - \varepsilon$. The

LR results for $I_1$, $I_2$, $I_3$ and $I_4(0)$ are implicit in the work of Fisher *et al* (1972). The parameter $b$ is considered to be very large and only the leading contributions are kept (Aharony 1976b). We obtain

$$I_1 = \int_{b^{-1}}^{1} q^{\sigma-1}\,\mathrm{d}q = \frac{1}{\sigma}(1-b^{-\sigma}) \tag{A1}$$

$$I_2 = \int_{b^{-1}}^{1} q^{d-2\sigma-1}\,\mathrm{d}q = \frac{1}{\varepsilon'}(b^{\varepsilon'}-1) \tag{A2}$$

$$I_3 = 2\int_{b^{-1}}^{1} p^{\sigma-1}\,\mathrm{d}p \int_{b^{-1}}^{1} q^{-1} J(p,q)\,\mathrm{d}q \tag{A3}$$

where

$$J(p,q) = q^{-\sigma} F(\tfrac{1}{2}\sigma, 1-\tfrac{1}{2}\sigma, \sigma; p^2/q^2) \qquad \text{if } q > p \tag{A4}$$

and $F(\alpha, \beta, \gamma; x)$ is the hypergeometric series. For $I_3$ we then obtain

$$I_3 = \ln(b)(\ln(b) + A(\sigma)) + \mathrm{O}(1) \tag{A5}$$

where calling $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$:

$$\begin{aligned} A(\sigma) &= \frac{2}{\sigma} + 2\sum_{n=1}^{\infty} \frac{(\frac{1}{2}\sigma)_n(1-\frac{1}{2}\sigma)_n}{(\sigma)_n}\frac{1}{n!}\left(\frac{1}{2n}+\frac{1}{2n+\sigma}\right) \\ &= \Psi(\sigma) + \Psi(1) - 2\Psi(\tfrac{1}{2}\sigma) \end{aligned} \tag{A6}$$

and $\Psi(x)$ is the digamma function (Gradshteyn and Ryzhik 1965, Erdelyi *et al* 1954).

To evaluate $I_5$ we start by evaluating

$$I_4(\boldsymbol{k}) = (2\pi)^{-d} K_d^{-1} \int_{>} \mathrm{d}\boldsymbol{q}\, q^{-\sigma} \int_{b^{-1}}^{1} p^{\sigma-1} J(p, |\boldsymbol{q}+\boldsymbol{k}|)\,\mathrm{d}p \tag{A7}$$

with $J(p,q)$ defined in equation (A4). The first integral over the variable $p$ is easily expressed as a series in powers of $|\boldsymbol{q}+\boldsymbol{k}|$ plus a logarithmic term. After differentiating twice with respect to $|\boldsymbol{k}|$ and integrating over the angle between $\boldsymbol{k}$ and $\boldsymbol{q}$ we obtain

$$I_4(0) = \frac{2}{\sigma}\left(\frac{1}{\sigma} + \sum_{n=1}^{\infty} \frac{(\frac{1}{2}\sigma)_n(1-\frac{1}{2}\sigma)_n}{(\sigma)_n}\frac{1}{n!}\frac{1}{2n+\sigma}\right) \tag{A8}$$

$$\begin{aligned} I_5(\sigma) = -\frac{1}{2}\left(\frac{\partial^2 I_4(k)}{\partial k^2}\right)_{k=0} &= \frac{(\sigma-1)}{2\sigma(2-\sigma)}(b^{(2-\sigma)}-1) \\ &+ b^{(2-\sigma)}\frac{1}{8}\sum_{n=0} \frac{(\frac{1}{2}\sigma)_n(1-\frac{1}{2}\sigma)_n}{(\sigma)_n}\frac{1}{n!}\left(\frac{2}{\sigma}-\frac{1}{n+1}\right) + \mathrm{O}(1). \end{aligned} \tag{A9}$$

Then for $\sigma < 2$ the leading terms in $I_5(\sigma)$ are $\mathrm{O}(b^{2-\sigma})$ while for $\sigma = 2$

$$I_5(\sigma=2) = \tfrac{1}{4}\ln(b) + \mathrm{O}(1). \tag{A10}$$

## References


Aharony A 1975 *Phys. Rev.* B **12** 1049–56

—— 1976a *Phys. Rev.* B **13** 2092–8

—— 1976b *Phase Transitions and Critical Phenomena* vol 6 ed C Domb and M S Green (New York: Academic) ch 6, pp 357–424

Baker G A Jr 1972 *Phys. Rev.* B **5** 2622–33
Bleher P M and Sinai Ya G 1975 *Commun. Math. Phys.* **45** 247–78
Emery V J 1975 *Phys. Rev.* B **11** 239–47
Erdelyi A, Magnus W, Oberhettinger F and Tricomi F 1953 *Higher Transcendental Functions* vol 1 (New York: McGraw-Hill) ch 1, p 8, equation (3)
Fisher M E 1974 *Rev. Mod. Phys.* **46** 597–616
Fisher M E, Ma S K and Nickel B G 1972 *Phys. Rev. Lett.* **29** 917–20
Gradshteyn I S and Ryzhik I M 1965 *Tables of Integrals, Series and Products* (New York: Academic)
Grinstein G and Luther A 1976 *Phys. Rev.* B **13** 1329–43
Jayaprakash C and Katz H J 1977 *Phys. Rev.* B **16** 3987–90
Khmelnitskii D E 1975 *Zh. Eksp. Teor. Fiz.* **68** 1960–8 (Transl 1976 *Sov. Phys.-JETP* **41** 981–4)
Kim D and Thompson C J 1977 *J. Phys. A: Math. Gen.* **10** 1579–98
Ma S K 1973 *Phys. Rev.* A **7** 2172–87
Sak J 1973 *Phys. Rev.* B **8** 281–5
Theumann A 1980a *Phys. Rev.* B **21** 2984–90
—— 1980b *Phys. Rev.* B **22** 5441
Theumann W K 1975 *Physics* **83** A 85–120
Yamazaki Y 1978 *Physica* **90** A 547–60
Yamazaki Y and Suzuki M 1977 *Prog. Theor. Phys.* **57** 1886–99